BANDGAPS IN OCTET TRUSS LATTICES

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ABSTRACT

A finite element calculation based on three-dimensional Timoshenko beams is coupled to a Floquet -Bloch analysis, enabling the determination of the frequency dispersion curves for infinite three-dimensional periodic lattices. The technique is applied to the octet truss, and it is shown that for certain values of the aspect ratio of the struts in the unit cells, complete bandgaps exist.

Keywords: Lattice materials, vibration, bandgaps.

BACKGROUND

Acoustic metamaterials have been the subject of much recent study. The wave propagation and bandgap characteristics of periodic structures have been analysed using a variety of methods, primarily the finite difference time domain (FDTD) method [1, 2], the plane wave expansion (PWE) method [3], and the multiple scattering (MS) method [4, 5]. Other methods include a lumped mass formuation [6], and a transfer matrix method [7]. These methods have generally been applied to the analysis of phononic crystals, which consist of periodic inclusions embedded in a continuum. Periodic lattices composed of beam-like members are more suited to analysis by a finite element method, as was done in [8] for two-dimensional lattices.

The geometry of any periodic lattice can be fully described by specifying the unit cell and the periodicity vectors, which define how the unit cell should be tessellated to obtain the periodic lattice. The vectors that define the periodicity of the lattice are called the direct basis vectors, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. They specify the position of cells in the lattice, in relation to a reference cell. Each integer tuple (n_1, n_2, n_3) identifies a cell in the lattice, with position $n_1\vec{e}_1+n_2\vec{e}_2+n_3\vec{e}_3$ in relation to the reference cell. The position vector of any point $\vec{r}$ in the (n_1, n_2, n_3)-cell can be written as

$$\vec{r} = \vec{r}_r + n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$$

where $\vec{r}_r$ is the position vector of the corresponding point in the reference cell. The lattice examined here is the octet lattice, shown in figure 1(a), which is created by stacking tetrahedral unit cells [9, 10].

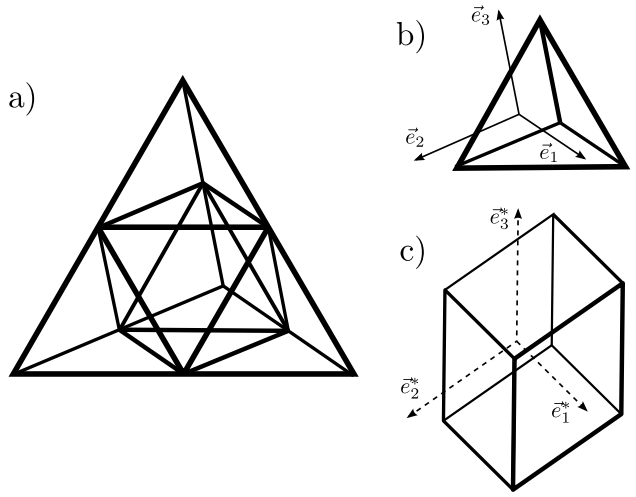

Figure 1: a) Octet lattice; b) Unit cell and direct basis vectors; c) First Brillouin zone and reciprocal basis vectors

The classical equation describing plane wave motion with wave vector $\vec{k}$, frequency ω, amplitude $\mathbf{A}$ at a point $\vec{r}$ and at time t is

$$\mathbf{q}(\vec{r}) = \mathbf{A}e^{i(\vec{k}\cdot\vec{r}-\omega t)}.$$

In a lattice, the frequency of wave propagation ω is periodic with respect to the wave vector $\vec{k}$ [11]. The frequency of wave propagation ω will be the same for two wave vectors $\vec{k}$ and $\vec{k}'$ related by:

$$\vec{k}' = \vec{k} + m_1\vec{e}_1^* + m_2\vec{e}_2^* + m_3\vec{e}_3^* \text{ where } m_1, m_2, m_3 \in \mathbb{Z}.$$

The reciprocal vectors $\vec{e}_1^*$, $\vec{e}_2^*$, $\vec{e}_3^*$ are the basis vectors that define the periodicity of the frequency and the reciprocal lattice. The basic unit cell of that lattice is known as the first Brillouin zone [11]. For any wave vector in the real lattice, a corresponding wave vector can be found in the first Brillouin zone with the same frequencies of propagation. The full frequency response of the lattice is characterised by its frequency response to wave vectors in the first Brillouin zone. Note that due to the periodicity of the frequency, *any* basic unit cell with the reciprocal vectors as the basis can be used as the first Brillouin zone. The simplest choice of the first Brillouin zone is the parallelpiped whose sides are the reciprocal basis vectors.

APPROACH

A. Finite Element Model

To obtain the equation of motion for the reference cell of a lattice, the reference cell is discretised into a network of Timoshenko elements [12]. For each Timoshenko element, the kinetic and potential energies are expressed in terms of the displacements at the nodes of the element. The total kinetic and potential energies in the reference cell can then be expressed as the sum of the element energies, and put in terms of the displacements of the nodes. These nodal displacements form generalised co-ordinates, and allow for the Euler-Lagrange equation to be used to derive the equation of motion for the reference cell.

A three-dimensional Timoshenko element has six degrees of freedom per node – three in translation (u, v, w) and three in rotation (ϕ, ψ, θ). The element has displacement field:

$$\mathbf{u} = \begin{bmatrix} u - y\theta + z\psi \\ v - z\phi \\ w + y\phi \end{bmatrix}$$

where y, z are the undeformed distances of the point of interest from the beam centreline, and u, v, w are the translations at this point.

The kinetic energy of element l is:

$$\begin{aligned} T_l &= \frac{1}{2}\int_0^L A\rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) + I_z\rho\dot{\theta}^2 \\ &\quad + I_y\rho\dot{\psi}^2 + (I_z + I_y)\rho\dot{\phi}^2 \,\mathrm{d}x, \end{aligned}$$

which is written as:

$$T_l = \frac{1}{2}\dot{\mathbf{q}}_l^T\mathbf{m}_l\dot{\mathbf{q}}_l.$$

Similarly, the total strain potential energy U_l for the element l is:

$$\begin{aligned} U_l &= \sum_{ij}\frac{1}{2}q_iq_j\int_0^L\Big[E\left(Aa_i'a_j' + I_ye_i'e_j' + I_zf_i'f_j'\right) \\ &\quad + (I_z + I_y)\kappa G d_i'd_j' \\ &\quad + \kappa GA\left((b_i' - f_i)(b_j' - f_j) + (c_i' + e_i)(c_j' + e_j)\right)\Big]\mathrm{d}x \end{aligned}$$

In matrix form

$$U_l = \frac{1}{2}\mathbf{q}_l^T\mathbf{k}_l\mathbf{q}_l$$

The kinetic and potential energies of all the elements are summed to give to the total energy of the reference cell by transforming the nodal displacements of each element to a global co-ordinate system, equating the nodal displacements of connected elements, and assembling all the nodal displacements into a single vector $\mathbf{q}$. The mass and stiffness matrices of each of the elements can be added to give global mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$.

B. Floquet - Bloch Principles

Bloch's theorem provides boundary conditions that enforce a plane wave solution in an infinite lattice, which is a special case of the wave equation in a periodic medium. The classical equation describing plane wave motion with wave vector $\vec{k}$, frequency ω, amplitude $\mathbf{A}$ at a point $\vec{r}$ and at time t is

$$\mathbf{q}(\vec{r}) = \mathbf{A}e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

The displacement at a point $\vec{r}$ in the three-dimensional lattice which corresponds to a point in the reference cell $\vec{r}_j$ is:

$$\mathbf{q}(\vec{r}) = \mathbf{q}(\vec{r}_j)e^{i\vec{k}\cdot(\vec{r} - \vec{r}_j)}$$

For a three-dimensional periodic structure with basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, $\vec{r}$ and $\vec{r}_j$ are related in the following manner:

$$\mathbf{q}(\vec{r}) = \mathbf{q}(\vec{r}_j)e^{n_1k_1 + n_2k_2 + n_3k_3}$$

This result is known as Bloch's theorem [8]. k_1, k_2, and k_3 are phase constants, determined by the wave vector and the basis vectors of the lattice. A transformation matrix $\mathbf{T}$, which is dependent on the wave vector $\vec{k}$, relates the complete set of generalised co-ordinates $\mathbf{q}$ to the reduced set $\tilde{\mathbf{q}}$:

$$\mathbf{q} = \mathbf{T}\tilde{\mathbf{q}},$$

Using the above result,

$$\tilde{\mathbf{M}}\ddot{\tilde{\mathbf{q}}} + \tilde{\mathbf{K}}\tilde{\mathbf{q}} = \mathbf{T}^H\mathbf{f}, \tag{1}$$

where $()^H$ denotes the the conjugate transpose, and $\tilde{\mathbf{K}} = \mathbf{T}^H\mathbf{K}\mathbf{T}$ and $\tilde{\mathbf{M}} = \mathbf{T}^H\mathbf{M}\mathbf{T}$. This equation is homogeneous because $\mathbf{T}^H\mathbf{f}$ can be shown to be zero. Thus:

$$\tilde{\mathbf{K}}\mathbf{A} = \omega^2\tilde{\mathbf{M}}\mathbf{A} \tag{2}$$

Since $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{M}}$ are Hermitian, (2) forms a generalised Hermitian eigenvalue problem which can be be solved for the eigenvalue ω^2 and the eigenvector $\mathbf{A}$ for a particular wave vector (which determines $\mathbf{T}$ and hence $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$) .

RESULTS

Using this method, dispersion curves for the octet truss were generated. Since the frequency of wave propagation in a periodic structure is periodic with respect to the wave vector [11], only the basic unit of this periodicity needs to be examined to calculate the complete frequency response of the octet truss. The wave vectors used in the eigenvalue problem (2) are restricted to the first Brillouin zone [11], which forms the basic unit of periodicity of the frequency in the space of wave vectors. This study examines wave vectors from the entire first Brillouin zone. The methods employed are applicable to lattice geometries for which band extrema do not occur on the edges of the irreducible Brillouin zone, and to lattice geometries with non-symmetric reference cells.

Since the first Brillouin zone for the octet lattice is a three-dimensional region, visualising the possible frequencies for each wave vector in this region would require a four-dimensional graph. Instead, the results are shown in dispersion curves that plot the frequency of wave propagation against a single component of the wave vector, for all wave vectors in the first Brillouin zone. This is equivalent to projecting the four-dimensional frequency surface onto a single plane in the space spanned by k_1, k_2, k_3, and ω. Any frequency bandgaps in such dispersion plots are immediately apparent, and are marked by white space.

Figure 2 and figure 3 plot the dispersion characteristics for radius-to-length ratios of 10 and 50, respectively. The material properties used were Young's modulus $E = 200$ GPa, density $\rho = 1$ Mg m^{-3}, and Poisson's ratio $\nu = 0.3$. To generate these figures, the three-dimensional Brillouin zone was discretised into a network of points. The eigenproblem (2) was solved for each value of $\vec{k}$, and the resulting eigenvalues ω were plotted against a component of $\vec{k}$. Each point in the figures represents a frequency of wave propagation for a given wave vector. The vertical lines in the plots are artifacts of the discretisation scheme used to extract wave vectors from the Brillouin zone.

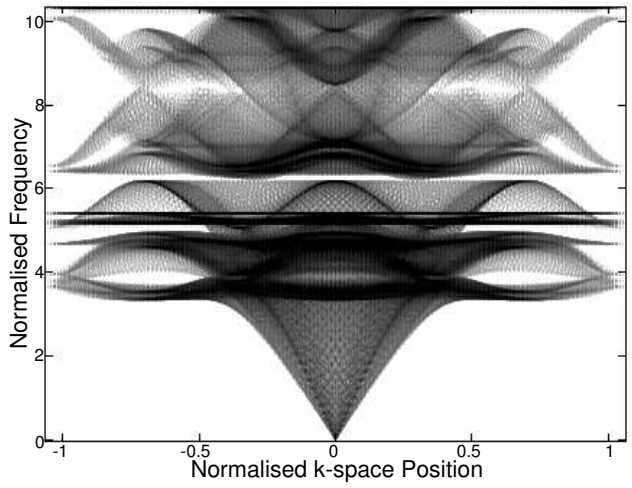

Figure 2: Dispersion curves for an octet lattice with a radius-to-length ratio of 10.

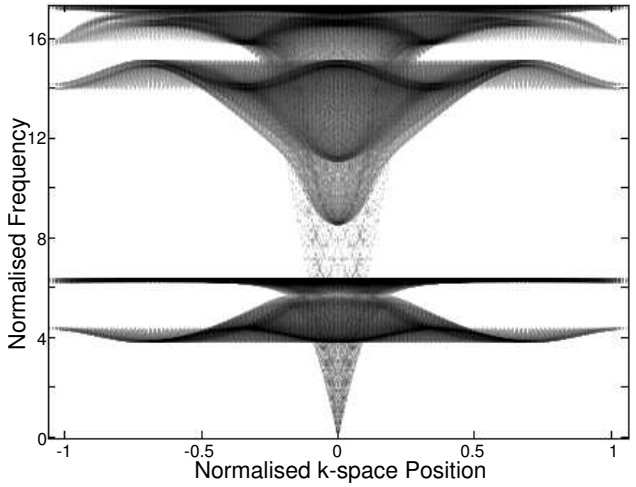

Figure 3: Dispersion curves for an octet lattice with a radius-to-length ratio of 50.

As is evident from the figures, the band structure of the octet lattice depends heavily on the radius-to-length ratio. Varying the elasticity and density of the lattice material has no effect on the band structure; instead this variation scales all frequencies by a constant factor. As can be seen from Figure 2, the octet lattice exhibits a thin but complete bandgap centred at 780 rad/s for a radius-to-length ratio of 10. The lattice does not have any complete frequency bandgaps for a radius-to-length ratio of 50. However, several partial bandgaps can be seen in Figure 3.

CONCLUDING COMMENTS

The octet truss has the important characteristic that, for certain values of the aspect ratio of the struts in the unit cell, complete wave propagation bandgaps exist. Moreover, by selecting the appropriate constituent materials, the location of the bandgap can be adjusted. Hence the octet truss is a candidate three-dimensional lattice material for applications which require frequency or vibration isolation.

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